Faculty of Science, Technology, Engineering and Mathematics
MST210 Mathematical methods, models and modelling

MST210

TMA 06 2019J

Covers Units 15, 16 and 17 **Cut-off date 25 March 2020**

The module website gives details of how to submit assignments for this module. If you have a disability that makes it difficult for you to attempt any of the questions, then please contact your Student Support Team or your tutor for advice.

The marks allocated to each part of each question are indicated in brackets in the margin.

You may use Maxima to check answers, but it should not be used otherwise to answer TMA questions except where explicitly indicated. TMA answers must show full working to attract full marks.

In order to encourage you to present your solutions to the TMA questions in a good mathematical style, your tutor will comment on how you:

- use correct mathematical notation
- define any symbols that you introduce in formulating and solving a problem
- give references for standard formulas and derivations
- include comments and explanations within your mathematics
- explicitly state results and conclusions, giving answers to an appropriate degree of accuracy and interpreting answers in the context of the question
- draw diagrams and graphs.

These features are seen as being essential to complementing your mathematical skills. Your tutor will make comments on how well you achieve these objectives and give you guidance on how to satisfy them.

Five of the marks for this TMA will be allocated to the way you write your solutions. It is expected that most students will receive the majority of these presentation marks; such marks are included in TMAs to encourage, and emphasise the need for, thinking about how you present your mathematics.

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Question 1 (Unit 15) – 15 marks

The temperature T of a point with coordinates (x, y) lying in a region of the (x, y)-plane is given by the function

$$T(x, y) = -3x^2 - 2xy - 6x - y^2 - y.$$

(a) Determine $\mathbf{grad}\, T$ and evaluate it at the point $(-1, 1)$. [3]

(b) What is the value of the derivative of T at the point $(-1, 1)$ in the direction of the vector $\mathbf{d} = \mathbf{i} + \mathbf{j}$? [3]

(c) Find the maximum rate of change in temperature at the point $(-1, 1)$. What is its direction? [2]

(d) Show that the temperature is 3 at the point $(-1, 1)$. At this point, determine the equation of the tangent to the $T = 3$ contour of temperature, in the form $y = mx + c$, where m and c are constants that you should determine. [7]

Question 2 (Unit 15) – 18 marks

Consider the three-dimensional scalar field

$$h(x, y, z) = \frac{1}{x^2 + y^2 + z^2} + z.$$

(a) Find an expression for the scalar field h in cylindrical coordinates. [2]

(b) Determine the gradient $\mathbf{grad}\, h$ in cylindrical coordinates. [5]

(c) Find an expression for the scalar field h in spherical coordinates. [2]

(d) Determine the gradient $\mathbf{grad}\, h$ and the value of $|\mathbf{grad}\, h|$ in spherical coordinates. [9]

Question 3 (Unit 16) – 8 marks

Evaluate the scalar line integral of the vector field

$$\mathbf{F}(x, y, z) = -y\mathbf{i} + x\mathbf{j} + z\mathbf{k}$$

around the closed circular path C of radius 1 centred at the point $(-2, 0, 3)$ defined by the parametric equations

$$x = \cos t - 2, \quad y = \sin t, \quad z = 3, \quad \text{for } 0 \leq t \leq 2\pi.$$

Is the vector field $\mathbf{F}$ conservative? [8]

Question 4 (Unit 16) – 14 marks

Consider the vector field

$$\mathbf{F}(x, y, z) = yz\mathbf{i} + (xz + 2yz^2 - 2)\mathbf{j} + (xy + 2y^2 z)\mathbf{k}.$$

(a) Calculate $\mathbf{curl}\, \mathbf{F}$ and state whether or not $\mathbf{F}$ is conservative. [5]

(b) Follow Procedure 1 on page 94 of Unit 16 to calculate a potential function U such that $\mathbf{F} = -\mathbf{grad}\, U$. [7]

(c) Using the expression for $U(x, y, z)$ that you found in part (b), calculate $-\mathbf{grad}\, U$ and confirm that $\mathbf{F} = -\mathbf{grad}\, U$. [2]

Question 5 (Unit 16) – 3 marks

A vector field $\mathbf{F}$ is expressed in cylindrical coordinates as

$$\mathbf{F}(\rho, \phi, z) = \frac{z + \sin\phi}{\rho}\,\mathbf{e}_\rho + \cos\phi\,\mathbf{e}_\phi + \sin z\,\mathbf{e}_z.$$

Determine $\operatorname{div}\mathbf{F}$. [3]

Question 6 (Unit 16) – 5 marks

A velocity field $\mathbf{v}$ is expressed in spherical coordinates as

$$\mathbf{v}(r, \theta, \phi) = r\sin\phi\sin(2\theta)\,\mathbf{e}_r + r\sin\phi\cos(2\theta)\,\mathbf{e}_\theta + r\cos\phi\cos\theta\,\mathbf{e}_\phi.$$

By calculating $\boldsymbol{\nabla} \times \mathbf{v}$, show that $\mathbf{v}$ is conservative. [5]

Question 7 (Unit 17) – 10 marks

A region in the (x, y)-plane is bounded by the curves

$$y = \frac{1}{8}\pi^2 - \frac{1}{2}x^2 \quad \text{and} \quad y = 3\cos x, \quad \text{for } -\pi/2 \le x \le \pi/2.$$

These curves meet at $x = \pm\pi/2$, but not at any other point within the interval $[-\pi/2, \pi/2]$.

(a) Sketch a diagram showing the region. Indicate a thin vertical strip within the region, and mark the values of y at its endpoints. [3]

(b) Use an area integral to find the area of the region. [7]

Question 8 (Unit 17) – 9 marks

A bullet-shaped object is formed from a cylinder with a curved top. The cylinder has radius a and is situated with its base on the (x, y)-plane, with the origin located at the centre of the base. The curved top of the bullet is given by the formula $z = 3a - x^2/a - y^2/a$ and the density is given by $D\left(\dfrac{\sqrt{x^2 + y^2}}{a} + 1\right)$, where a and D are constants.

Calculate the mass of the object. [9]

Question 9 (Unit 17) – 13 marks

A thick spherical shell occupies the region between two spheres of radii a and $2a$, both centred on the origin. The shell is made of a material with density $\rho = \dfrac{Ax^2z^2}{(x^2 + y^2 + z^2)^2}$, where A is a constant.

(a) Show that the density expressed in spherical coordinates (r, θ, ϕ) is

$$\rho = A\sin^2\theta\cos^2\phi\cos^2\theta.$$ [2]

(b) Hence, or otherwise, find the mass M of the shell by evaluating a suitable volume integral.

(*Hint*: To evaluate the integral

$$\int_0^\pi \sin^3\theta\cos^2\theta\,d\theta,$$

use the substitution $u = \cos\theta$.) [11]